PERIODIC SOLUTIONS OF QUASILINEAR AUTONOMOUS SYSTEMS WITH ONE DEGREE OF FREEDOM IN THE FORM OF INFINITE SERIES WITH FRACTIONAL POWERS OF THE PARAMETER

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The periodic solutions of quasilinear systems are usually represented as infinite series with integer powers of a small parameter [1,2]. The present work also deals with solutions in series, but the powers of the parameter are fractional.

1. Let us consider a quasilinear oscillating system of the form

$$\frac{d^2x}{dt^2} + k^2x = \mu f(x, \dot{x}, \mu) \tag{1.1}$$

The function $f(x, \dot{x}, \mu)$ is assumed to be analytic in all of its arguments in some region. The parameter μ is assumed to be small.

Since the system is autonomous, the solution of the generating equation (with $\mu = 0$) will be

$$x_0(t) = A_0 \cos kt$$

We assume that the initial conditions for the system (1.1) are given in the form

$$x(0) = A_0 + \beta, \qquad \dot{x}(0) = 0 \tag{1.2}$$

where β is a function of μ that vanishes when $\mu = 0$.

Let us assume that the region of analiticity of the function $f(x, \dot{x}, \mu)$ contains the generating solution $x_0(t)$. On the basis of known theorems [3], the solution $x(t, \beta, \mu)$ of Equation (1.1) will be analytic in t, and for small enough values of the parameter μ, also in μ and β. Let us express this solution in the form

$$x(t,\beta,\mu)=(A_0+\beta)\cos kt+\sum_{n=1}^{\infty}\left[C_n(t)+\frac{\partial C_n(t)}{\partial A_0}\beta+\frac{1}{2}\frac{\partial^2 C_n(t)}{\partial A_0^2}\beta^2+\ldots\right]\mu^n \tag{1.3}$$

We next introduce the notation

$$H_n(t)=\frac{1}{(n-1)!}\left(\frac{d^{n-1}f}{d\mu^{n-1}}\right)_{\beta=\mu=0}$$

where $H_1(t) = f(x_0, \dot{x}_0, 0)$. The formulas for the next three quantities $H_n(t)$ are given in [2].

It is not difficult to see that the coefficients $C_n(t)$ are determined by the equation

$$C_n(t)=\frac{1}{k}\int^{t} H_n(t_1)\sin k(t-t_1)\,dt_1 \tag{1.4}$$

The oscillation period of an autonomous system depends on the parameter μ and can be expressed in the form $T = T_0 + \alpha$, where $T_0 = 2\pi/k$, and α is some function of μ which vanishes when $\mu = 0$.

The condition for periodicity of the function $x(t, \beta, \mu)$, and of its first derivative with respect to t can be written as

$$x(T_0+\alpha,\beta,\mu)=A_0+\beta,\qquad \dot{x}(T_0+\alpha,\beta,\mu)=0 \tag{1.5}$$

The second one of these equations can be considered as an equation that determines α as an implicit function of β and μ. Since

$$\ddot{x}(T_0,0,0)=-k^2A_0$$

there exists a single-valued analytic function $\alpha(\beta, \mu)$ if $A_0 \neq 0$.

Bearing in mind that all partial derivatives of α with respect to β vanish when $\mu = 0$, we can express the function $\alpha(\beta, \mu)$ in the form

$$\alpha(\beta,\mu)=\sum_{n=1}^{\infty}\left(N_n+\frac{\partial N_n}{\partial A_0}\beta+\frac{1}{2}\frac{\partial^2 N_n}{\partial A_0^2}\beta^2+\ldots\right)\mu^n. \tag{1.6}$$

The calculation yields

$$N_1=\frac{1}{k^2A_0}\dot{C}_1(T_0),\quad N_2=\frac{1}{k^2A_0}\left[\dot{C}_2(T_0)+N_1\ddot{C}_1(T_0)\right]$$

$$N_3=\frac{1}{k^2A_0}\Big\{\dot{C}_3(T_0)+N_2\ddot{C}_1(T_0)+N_1\ddot{C}_2(T_0)+ \tag{1.7}$$

$$+\frac{1}{2}N_1^2\left[\dddot{C}_1(T_0)+\frac{1}{3}k^2\dot{C}_1(T_0)\right]\Big\}\ \text{и т. д.}$$

Substituting the value of α from (1.6) into the first equation of (1.5), we obtain

$$\sum_{n=1}^{\infty}\left(M_n+\frac{\partial M_n}{\partial A_0}\beta+\frac{1}{2}\frac{\partial^2 M_n}{\partial A_0^2}\beta^2+\ldots\right)\mu^n=0 \tag{1.8}$$

The quantities M_n are given by the equations

$$\begin{aligned} M_1 &= C_1(T_0), \qquad M_2 = C_2(T_0)+\frac{1}{2}N_1\dot{C}_1(T_0)\\ M_3 &= C_3(T_0)+N_2\dot{C}_1(T_0)-\frac{1}{2}N_1^2\ddot{C}_1(T_0)\\ M_4 &= C_4(T_0)+N_3\dot{C}_1(T_0)+\frac{1}{2}k^2A_0N_2^2-\\ &-N_1\left[N_2\ddot{C}_1(T_0)+\frac{1}{3}N_1^2\dddot{C}_1(T_0)+\frac{1}{2}N_1\ddot{C}_2(T_0)\right]\quad \text{etc.} \end{aligned} \tag{1.9}$$

2. Equations (1.8) determine an implicit function $\beta=\beta(\mu)$. Dividing out μ, and taking into account that $\beta(0)=0$, we obtain

$$M_1=C_1(T_0)=0 \tag{2.1}$$

Let us suppose that $C_1(T_0)$ is not identically zero. Then Equation (2.1) will be the equation for the amplitudes A_0 of the generating solution.

If $C_1(T_0)\equiv 0$, then all the derivatives of $C_1(T_0)$ with respect to A_0 will also be equal to zero. The equation for the amplitudes would then be given by $M_2=0$ under the condition that M_2 be not identically zero, and so on.

Let us write out (1.8) in its expanded form by grouping its terms as homogeneous polynomials in β and μ:

$$\begin{aligned}\Phi(\beta,\mu) &= \frac{\partial C_1}{\partial A_0}\beta+M_2\mu+\frac{1}{2}\frac{\partial^2 C_1}{\partial A_0}\beta^2+\frac{\partial M_2}{\partial A_0}\beta\mu+M_3\mu^2+\\ &+\frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3}\beta^3+\frac{1}{2}\frac{\partial^2 M_2}{\partial A_0^2}\beta^2\mu+\frac{\partial M_3}{\partial A_0}\beta\mu^2+M_4\mu^3+\ldots=0\end{aligned} \tag{2.2}$$

When $\mu=0$, we have

$$\Phi(\beta,0)=\frac{\partial C_1}{\partial A_0}\beta+\frac{1}{2}\frac{\partial^2 C_1}{\partial A_0^2}\beta^2+\frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3}\beta^3+\ldots$$

According to a theorem of Weierstrass on implicit functions [3,4], the number of roots of Equation (2.2), i.e. the number of implicit functions $\beta(\mu)$ determined by this equation, is equal to the lowest exponent of the expansion of $\Phi(\beta,0)$ in powers of β.

Hence, in the given case, the mentioned number of roots $\beta(\mu)$ is equal to the multiplicity of the root of the amplitudal equation (2.1).

Suppose that the multiplicity of the considered root A_0 is m. Then, under the assumptions made, all the m roots of Equation (2.2) can be expanded into convergent series of the form

$$\beta = \sum_{n=1}^{\infty} A_{n/k}\mu^{n/k} \tag{2.3}$$

where k can be equal to any integer from 1 to m inclusive. Hereby, there may exist simultaneously an expansion $\beta(\mu)$ in terms of fractional powers of the parameter μ, but the sum of the various k cannot exceed the number m.

Let us consider some of the more simple cases.

1. The quantity A_0 is a simple root of Equation (2.1). In this case, as is known, there exists a unique expansion of the form (2.3) when $k = 1$.

Let us introduce the notation

$$\begin{aligned} P_n(A_1) &= \frac{1}{n!}\frac{\partial^n C_1}{\partial A_0{}^n} A_1{}^n + \frac{1}{(n-1)!}\frac{\partial^{n-1} M_2}{\partial A_0{}^{n-1}} A_1{}^{n-1} + \ldots + M_{n+1} \\ Q_n(A_2) &= \frac{1}{n!}\frac{\partial^n P_2}{\partial A_1{}^n} A_2{}^n + \frac{1}{(n-1)!}\frac{\partial^{n-1} P_3}{\partial A_1{}^{n-1}} A_2{}^{n-1} + \ldots + P_{n+2} \end{aligned} \tag{2.4}$$

The coefficients A_n are determined by means of an infinite system of linear equations

$$\begin{gathered} P_1(A_1) = \frac{\partial C_1}{\partial A_0} A_1 + M_2 = 0, \qquad \frac{\partial C_1}{\partial A_0} A_2 + P_2 = 0 \\ \frac{\partial C_1}{\partial A_0} A_3 + \frac{\partial P_2}{\partial A_1} A_2 + P_3 = 0 \\ \frac{\partial C_1}{\partial A_0} A_4 + \frac{\partial P_2}{\partial A_1} A_3 + \frac{1}{2}\frac{\partial^2 C_1}{\partial A_0{}^2} A_2{}^2 + \frac{\partial P_3}{\partial A_1} A_2 + P_4 = 0 \qquad \text{etc.} \end{gathered} \tag{2.5}$$

2. The quantity A_0 is a double root of Equation (2.1). Then

$$\frac{\partial C_1}{\partial A_0} = 0, \qquad \frac{\partial^2 C_1}{\partial A_0{}^2} \neq 0$$

Equation (2.2) has two roots $\beta = \beta(\mu)$ in this case. The sum of the denominators of the exponents of μ in the various types of expansions in each of the cases considered cannot exceed 2. Hence, we can have two types of expansions: either in integer powers of μ or in powers of $\mu^{1/2}$.

2.1. If $M_2 \neq 0$, then the expansion of β will have the form (2.3) with

$k = 2$. Let us introduce the notation

$$S_n(A_{1/2}) = \frac{1}{n!}\frac{\partial^n C_1}{\partial A_0^n} A_{1/2}{}^n + \frac{1}{(n-2)!}\frac{\partial^{n-2} M_2}{\partial A_0^{n-2}} A_{1/2}{}^{n-2} + \ldots \qquad (2.6)$$

We now obtain the following equations for the determination of the coefficients $A_{n/2}$:

$$S_2(A_{1/2}) = \frac{1}{2}\frac{\partial^2 C_1}{\partial A_0^2} A_{1/2}{}^2 + M_2 = 0, \qquad \frac{\partial S_2}{\partial A_{1/2}} A_1 + S_3 = 0$$

$$\frac{\partial S_2}{\partial A_{1/2}} A_{3/2} + \frac{1}{2}\frac{\partial^2 S_2}{\partial A_{1/2}{}^2} A_1^2 + \frac{\partial S_3}{\partial A_{1/2}} A_1 + S_4 = 0$$

$$\frac{\partial S_2}{\partial A_{1/2}} A_2 + \left(\frac{\partial^2 S_2}{\partial A_{1/2}{}^2} A_1 + \frac{\partial S_3}{\partial A_{1/2}}\right) A_{3/2} + \frac{1}{2}\frac{\partial^2 S_3}{\partial A_{1/2}{}^2} A_1^2 + \frac{\partial S_4}{\partial A_{1/2}} A_1 + S_5 = 0 \quad \text{etc.}$$

If the roots of the first one of these equations are real, then we have two expansions for β, whereby the remaining coefficients $A_{n/2}$ are determined successively by means of an infinite system of linear equations.

2.2. If $M_2 = 0$, then $A_{1/2} = 0$, and the coefficient A_1 is determined by means of a quadratic equation

$$P_2(A_1) = \frac{1}{2}\frac{\partial^2 C_1}{\partial A_0^2} A_1^2 + \frac{\partial M_2}{\partial A_0} A_1 + M_3 = 0 \qquad (2.7)$$

For the further analysis we transform Equation (2.2) with the aid of the substitution

$$\beta = (\gamma + A_1)\mu \qquad (2.8)$$

Taking into account (2.7), we obtain, after division by μ^2, the following equation:

$$\frac{\partial P_2}{\partial A_1}\gamma + P_3\mu + \frac{1}{2}\frac{\partial^2 C_1}{\partial A_0^2}\gamma^2 + \frac{\partial P_3}{\partial A_1}\gamma\mu + P_4\mu^2 + \frac{1}{2}\frac{\partial^2 P_3}{\partial A_1^2}\gamma^2\mu + \frac{\partial P_4}{\partial A_1}\gamma\mu^2 + P_5\mu^3 + \ldots = 0 \qquad (2.9)$$

a) The roots of Equation (2.7) are simple. In this case we have the expansion (2.3) with $k = 1$. The coefficients A_n are found from the system of equations (2.5).

b) The roots of Equation (2.7) are multiple roots, but $P_3 \neq 0$. We now have an expansion of the type (2.3) with $k = 2$. The equations for the determination of the coefficients $A_{n/2}$ take the form

$$\frac{1}{2}\frac{\partial^2 C_1}{\partial A_0^2} A_{3/2}{}^2 + P_3 = 0$$

$$\left(\frac{\partial^2 C_1}{\partial A_0^2} A_2 + \frac{\partial P_3}{\partial A_1}\right) A_{3/2} = 0$$

$$\frac{\partial^2 C_1}{\partial A_0^2} A_{3/2} A_{5/2} + \frac{1}{2} \frac{\partial^2 P_3}{\partial A_1^2} A_{3/2}^2 + Q_2 = 0 \qquad \text{etc.}$$

c) The roots of Equation (2.7) are multiple roots, and $P_3 = 0$. Then $A_{3/2} = 0$, and the coefficient A_2 is determined by means of the quadratic equation

$$Q_2(A_2) = \frac{1}{2} \frac{\partial^2 C_1}{\partial A_0^2} A_2^2 + \frac{\partial P_3}{\partial A_1} A_2 + P_4 = 0 \tag{2.10}$$

The following analysis is connected with the multiplicity of the roots of Equation (2.10), and is entirely analogous to the preceding one. If the roots are simple, then one has the expansion (2.3) with $k = 1$. If the roots are multiple roots, but $Q_3 \neq 0$, then we obtain expansion (2.3) with $k = 2$. In case of multiple roots with $Q_3 = 0$, the analysis can be reduced to the consideration of the roots of a quadratic equation for A_3, and so on.

The formulas are considerably simpler if some of the terms in Equation (2.2) happen to be zero. For example, if $M_3 = 0$, and $\partial M_2 / \partial A_0 = 0$, Equation (2.7) has a double root $A_1 = 0$, and so on.

3. The quantity A_0 is a triple root of Equation (2.1)

$$\frac{\partial C_1}{\partial A_0} = \frac{\partial^2 C_1}{\partial A_0^2} = 0, \qquad \frac{\partial^3 C_1}{\partial A_0^3} \neq 0$$

In this case, Equation (2.2) has three roots $\beta = \beta(\mu)$. It is possible to have three expansions for β in powers of μ, $\mu^{1/2}$, and $\mu^{1/3}$. Taking into account the fact that the sum of the denominators in the exponents of μ cannot exceed three in each case, one can show that there can exist simultaneously expansions in powers of μ and of $\mu^{1/2}$.

3.1. If $M_2 \neq 0$, then the expansion for β will have the form (2.3) with $k = 3$.

Let us introduce the notation

$$U_n(A_{1/3}) = \frac{1}{n!} \frac{\partial^n C_1}{\partial A_0^n} A_{1/3}^{\ n} + \frac{1}{(n-3)!} \frac{\partial^{n-3} M_2}{\partial A_0^{n-3}} A_{1/3}^{\ n-3} + \ldots \tag{2.11}$$

For the determination of the coefficients $A_{n/2}$ we have the following equations

$$U_3(A_{1/3}) = \frac{1}{6} \frac{\partial^3 C_1}{\partial A_0^3} A_{1/3}^3 + M_2 = 0$$

$$\frac{\partial U_3}{\partial A_{1/3}} A_{2/3} + U_4 = 0$$

$$\frac{\partial U_3}{\partial A_{1/3}} A_1 + \frac{1}{2} \frac{\partial^2 U_3}{\partial A_{1/3}^2} A_{2/3}{}^3 + \frac{\partial U_4}{\partial A_{1/3}} A_{2/3} + U_5 = 0$$

$$\frac{\partial U_3}{\partial A_{1/3}} A_{4/3} + \frac{\partial U_4}{\partial A_{1/3}} A_1 + \frac{1}{6} \frac{\partial^3 U_3}{\partial A_{1/3}{}^3} A_{2/3}{}^2 + \frac{1}{2} \frac{\partial^2 U_4}{\partial A_{1/3}{}^2} A_{2/3}{}^2 + \frac{\partial U_5}{\partial A_{1/3}} A_{2/3} + U_6 = 0 \quad \text{etc.}$$

Since the first of these equations determines only one real value of $A_{1/3}$, and since the equations for the remaining coefficients are all linear, there exists only one expansion for β with real coefficients. This will always be the case in the sequel when one of the coefficients is determined from a binomial cubic equation.

3.2. Let $M_2 = 0$, but $\partial M_2 / \partial A_0 \neq 0$. We shall look for an expansion of β of the type (2.3) with $k = 2$. We obtain a system of equations for the coefficients $A_{n/2}$

$$S_3 (A_{1/2}) = \left(\frac{1}{6} \frac{\partial^3 C_1}{\partial A_0^3} A_{1/2}{}^2 + \frac{\partial M_2}{\partial A_0}\right) A_{1/2} = 0, \qquad \frac{\partial S_3}{\partial A_{1/2}} A_1 + S_4 = 0$$

$$\frac{\partial S_3}{\partial A_{1/2}} A_{3/2} + \frac{1}{2} \frac{\partial^2 S_3}{\partial A_{1/2}{}^2} A_1^2 + \frac{\partial S_4}{\partial A_{1/2}} A_1 + S_5 = 0 \qquad \text{etc.}$$

The first equation has two distinct roots and one zero root. To the first two roots there corresponds an expansion of β of the form (2.3) with $k = 2$. To the root $A_{1/2} = 0$ there corresponds an expansion of the type (2.3) with $k = 1$, while the coefficients A_n are determined by the system of equations (2.5).

3.3. Let $M_2 = 0$, $\partial M_2 / \partial A_0 = 0$, but $M_3 \neq 0$. In this case the expansion of β will have the form (2.3) with $k = 3$, but it will start with the term containing $\mu^{2/3}$. The equations for the coefficients $A_{n/3}$ are

$$\frac{1}{6} \frac{\partial^3 C_1}{\partial A_0^3} A_{2/3}{}^3 + M_3 = 0, \qquad \left(\frac{\partial^3 C_1}{\partial A_0^3} A_1 + \frac{\partial^2 M_2}{\partial A_0^2}\right) A_{2/3} = 0$$

$$\left(\frac{1}{2} \frac{\partial^3 C_1}{\partial A_0^3} A_{2/3} A_{4/3} + \frac{\partial P_3}{\partial A_1} + \frac{1}{24} \frac{\partial^4 C_1}{\partial A_0^4} A_{2/3}{}^3\right) A_{2/3} = 0 \qquad \text{etc.}$$

3.4. Let $M_2 = 0$, $\partial M_2 / \partial A_0 = 0$, and $M_3 = 0$. In this case the coefficient A_1 is determined by the cubic equation

$$P_3 (A_1) = \frac{1}{6} \frac{\partial^3 C_1}{\partial A_0^3} A_1^3 + \frac{1}{2} \frac{\partial^2 M_2}{\partial A_0^2} A_1^2 + \frac{\partial M_3}{\partial A_0} A_1 + M_4 = 0 \tag{2.12}$$

Let us transform Equation (2.2) with the aid of the substitution (2.8). After division by μ^3, we obtain

$$\frac{\partial P_3}{\partial A_1}\gamma + P_4\mu + \frac{1}{2}\frac{\partial^2 P_3}{\partial A_1^2}\gamma^2 + \frac{\partial P_4}{\partial A_1}\gamma\mu + P_5\mu^2 + \\ + \frac{1}{6}\frac{\partial^3 P_3}{\partial A_1^3}\gamma^3 + \frac{1}{2}\frac{\partial^2 P_4}{\partial A_1^2}\gamma^2\mu + \frac{\partial P_5}{\partial A_1}\gamma\mu^2 + P_6\mu^3 + \ldots = 0 \qquad (2.13)$$

If the roots of (2.12) are simple, then the expansion for β will have the form (2.3) with $k = 1$. The equations for the remaining coefficients will be

$$\frac{\partial P_3}{\partial A_1}A_2 + P_4 = 0, \qquad \frac{\partial P_3}{\partial A_1}A_3 + Q_3 = 0 \qquad \text{etc.}$$

Depending upon the number of real roots of Equation (2.12), there will exist in the given case either one or three expansions for β with real coefficients.

3.5. a) Suppose that among the roots of Equation (2.12) there is a double root

$$\frac{\partial P_3}{\partial A_1} = 0, \qquad P_4 \neq 0$$

For this root the expansion of β will have the form (2.3) with $k = 2$. The equations for the determination of the coefficients $A_{n/2}$ will be

$$\frac{1}{2}\frac{\partial^2 P_3}{\partial A_1^2}A_{3/2}{}^2 + P_4 = 0, \qquad \left(\frac{\partial^2 P_3}{\partial A_1^2}\; A_2 + \frac{\partial P_4}{\partial A_1} + \frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3}A_{3/2}{}^2\right)A_{3/2} = 0 \qquad \text{etc.}$$

If the first one of these equations has a real root, then there will exist three expansions for β of the type (2.3); one with $k = 1$, and two with $k = 2$.

b) If in the preceding case $P_4 = 0$, then the coefficient A_2 is determined by the quadratic equation

$$Q_3(A_2) = \frac{1}{2}\frac{\partial^2 P_3}{\partial A_1^2}A_2^2 + \frac{\partial P_4}{\partial A_1}A_2 + P_5 = 0 \qquad (2.14)$$

The form of the expansion of β will depend on the multiplicity of the roots of this equation. The analysis of the various possible cases is analogous to the analysis for the case 2.

3.6. a) Suppose, finally, that the roots of Equation (2.12) are triple roots,

$$\frac{\partial P_3}{\partial A_1} = \frac{\partial^2 P_3}{\partial A_1^2} = 0, \qquad P_4 \neq 0$$

The expansion for β will have the form (2.3) with $k = 3$. The equations for the coefficients $A_{n/3}$ will be

$$\frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3}A_{4/3}^3 + P_4 = 0$$

$$\left(\frac{1}{2}\frac{\partial^3 C_1}{\partial A_0^3}A_{4/3}A_{5/3} + \frac{\partial P_4}{\partial A_1}\right)A_{4/3} = 0$$

$$\frac{1}{2}\frac{\partial^3 C_1}{\partial A_0^3}(A_{4/3}^2 A_2 + A_{5/3}^2 A_{4/3}) + \frac{\partial P_4}{\partial A_1}A_{5/3} + \frac{1}{2}\frac{\partial^2 P_4}{\partial A_1^2}A_{4/3}^2 = 0 \qquad \text{etc.}$$

b) If in the preceding case $P_4 = 0$, but $\partial P/\partial A_1 \neq 0$, then there will exist an expansion for β of the type (2.3) with $k = 2$. The equation for the coefficient $A_{3/2}$ will be

$$\left(\frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3}A_{3/2}^2 + \frac{\partial P_4}{\partial A_1}\right)A_{3/2} = 0$$

For the nonzero real roots of this equation we have the following relations:

$$\left(\frac{1}{2}\frac{\partial^3 C_1}{\partial A_0^3}A_{3/2}^2 + \frac{\partial P_4}{\partial A_1}\right)A_2 + \frac{1}{2}\frac{\partial^2 P_4}{\partial A_1^2}A_{3/2}^2 + P_5 = 0 \quad \text{etc.}$$

For the zero root $A_{3/2} = 0$, we obtain an expansion of the form (2.3) with $k = 1$. The equations for the coefficients A_n will be

$$\frac{\partial P_4}{\partial A_1}A_2 + P_5 = 0, \qquad \frac{\partial P_4}{\partial A_1}A_3 + Q_4 = 0$$

$$\frac{\partial P_4}{\partial A_1}A_4 + \frac{\partial Q_4}{\partial A_2}A_3 + Q_5 = 0 \quad \text{etc.}$$

c) Let the roots be triple roots. $P_4 = 0$, $\partial P_4/\partial A_1 = 0$, but $P_5 \neq 0$. Then we have an expansion for β of the form (2.3) with $k = 3$. Hereby $A_{4/3} = 0$. The remaining equations for the coefficients $A_{n/3}$ will be

$$\frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3}A_{5/3}^3 + P_5 = 0, \qquad \left(\frac{\partial^3 C_1}{\partial A_0^3}A_2 + \frac{\partial^2 P_4}{\partial A_1^2}\right)A_{5/3}^2 = 0 \quad \text{etc.}$$

d) We have triple roots, $P_4 = 0$, $\partial P_4/\partial A_1 = 0$, and $P_5 = 0$. In this case the coefficient A_2 is determined by means of the cubic equation

$$Q_4(A_2) = \frac{1}{6}\frac{\partial^3 P_3}{\partial A_1^3}A_2^3 + \frac{1}{2}\frac{\partial^2 P_4}{\partial A_1^2}A_2^2 + \frac{\partial P_5}{\partial A_1}A_2 + P_6 = 0 \qquad (2.15)$$

The further analysis is connected with the multiplicity of the roots of this equation, and is entirely analogous to the preceding discussion.

The cases when the quantity A_0 is a multiple root of Equation (2.1) with a multiplicity higher than three, will not be considered in this work.

From what has been said it follows that the multiplicity of a root of the amplitudal equation (2.1) corresponds to the phenomenon of bifurcation of the generating solution. This bifurcation will not exist if only one root $\beta = \beta(\mu)$ is real, or if all the roots are equal. In the latter case, the expansion $\beta(\mu)$ will have the form (2.3) with $k = 1$.

3. It is easily seen that the form of the expansion of the period of the solution of Equation (1.1), and also of the solution itself, corresponds to the form of the expansion of $\beta(\mu)$.

Let us first consider the case when the expansion for β has the form (2.3) with $k = 2$. Then

$$\alpha = T_0 \sum_{n=1}^{\infty} h_{n/2}\mu^{n/2} \tag{3.1}$$

For the coefficients $h_{n/2}$ we have the formulas

$$h_{1/2} = 0, \qquad h_1 = \frac{1}{T_0} N_1, \qquad h_{3/2} = \frac{1}{T_0}\frac{\partial N_1}{\partial A_0} A_{1/2}$$
$$h_2 = \frac{1}{T_0}\left(\frac{1}{2}\frac{\partial^2 N_1}{\partial A_0^2} A_{1/2}^2 + \frac{\partial N_1}{\partial A_0} A_1 + N_2\right) \quad \text{etc.} \tag{3.2}$$

For the construction of the periodic solution of Equation (1.1) with a constant period we make the following change of variables:

$$t = \frac{\tau}{k}\,(1 + h_1\mu + h_{3/2}\mu^{3/2} + h_2\mu^2 + \ldots)$$

Then we obtain the solution in the form of a series in powers of $\mu^{1/2}$

$$x(\tau) = x_0(\tau) + \mu^{1/2}x_{1/2}(\tau) + \mu x_1(\tau) + \mu^{3/2}x_{3/2}(\tau) + \ldots \tag{3.3}$$

whose coefficients have the constant period 2π. These coefficients are given by the formulas

$$x_0(\tau) = A_0\cos\tau, \qquad x_{1/2}(\tau) = A_{1/2}\cos\tau, \qquad x_1(\tau) = A_1\cos\tau + C_1(\tau) - h_1A_0\tau\sin\tau \tag{3.4}$$
$$x_{3/2}(\tau) = A_{3/2}\cos\tau + A_{1/2}\frac{\partial C_1(\tau)}{\partial A_0} - (h_{3/2}A_0 + h_1A_{1/2})\,\tau\sin\tau$$
$$x_2(\tau) = A_2\cos\tau + C_2(\tau) + A_1\frac{\partial C_1(\tau)}{\partial A_0} + \frac{1}{2}A_{1/2}^2\frac{\partial^2 C_1(\tau)}{\partial A_0^2} + - h_1\tau\frac{\partial C_1(\tau)}{\partial\tau} -$$
$$- (h_2A_0 + h_{3/2}A_{1/2} + h_1A_1)\,\tau\sin\tau - \frac{1}{2}h_1^2A_0\tau^2\cos\tau \quad \text{etc.}$$

In these formulas it is first assumed that $C_n(\tau/k) = C^*_n(\tau)$, but the asterisk is then dropped. For the case when the expansion of $\beta(\mu)$ has the form (2.3), with $k = 3$, we obtain

$$h_{1/3} = 0, \qquad h_{2/3} = 0, \qquad h_1 = \frac{1}{T_0} N_1$$

$$h_{4/3}=\frac{1}{T_0}\frac{\partial N_1}{\partial A_0}A_{1/3},\qquad h_{5/3}=\frac{1}{T_0}\left(\frac{1}{2}\frac{\partial^2 N_1}{\partial A_0^2}A_{1/3}^2+\frac{\partial N_1}{\partial A_0}A_{2/3}\right) \tag{3.5}$$

$$h_2=\frac{1}{T_0}\left(\frac{1}{6}\frac{\partial^3 N_1}{\partial A_0^3}A_{1/3}^3+\frac{\partial^2 N_1}{\partial A_0^2}A_{1/3}A_{2/3}+\frac{\partial N_1}{\partial A_0}A_1+N_2\right)\quad \text{etc.}$$

In this case the expansion of the solution takes the following form after a change of variables:

$$x(\tau)=x_0(\tau)+\mu^{1/3}x_{1/3}(\tau)+\mu^{2/3}x_{2/3}(\tau)+\mu x_1(\tau)+\ldots \tag{3.6}$$

The first few coefficients of this expansion are given by

$$x_0(\tau)=A_0\cos\tau,\qquad x_{1/3}(\tau)=A_{1/3}\cos\tau,\qquad x_{2/3}(\tau)=A_{2/3}\cos\tau$$

$$x_1(\tau)=A_1\cos\tau+C_1(\tau)-h_1A_0\,\tau\sin\tau$$

$$x_{4/3}(\tau)=A_{4/3}\cos\tau+A_{1/3}\frac{\partial C_1(\tau)}{\partial A_0}-(h_{4/3}A_0+h_1A_{1/3})\,\tau\sin\tau$$

$$x_{5/3}(\tau)=A_{5/3}\cos\tau+A_{2/3}\frac{\partial C_1(\tau)}{\partial A_0}+\frac{1}{2}A_{1/3}^2\frac{\partial^2 C_1(\tau)}{\partial A_0^2}-$$

$$-(h_{5/3}A_0+h_{4/3}A_{1/3}+h_1A_{2/3})\,\tau\sin\tau$$

$$x_2(\tau)=A_2\cos\tau+C_2(\tau)+A_1\frac{\partial C_1(\tau)}{\partial A_0}+A_{1/3}A_{2/3}\frac{\partial^2 C_1(\tau)}{\partial A_0^2}+h_1\tau\frac{\partial C_1(\tau)}{\partial\tau}-$$

$$-(h_2A_0+h_{5/3}A_{1/3}+h_{4/3}A_{2/3}+h_1A_1)\,\tau\sin\tau-\frac{1}{2}h_1^2A_0\tau^2\cos\tau \tag{3.7}$$

For the case of the expansion of β in terms of integer powers of μ, the corresponding formulas are obtained by equating to zero all the coefficients $A_{n/k}$, and $h_{n/k}$, whose subscripts are not integers, in one of the formulas of one of the preceding cases.

The problem of the determination of the radius of convergence of the obtained series is not considered here.

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